Digraph Complexity Measures and Applications in Formal Language Theory

by

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Overview

1. Introduction and Motivation
2. Measuring Complexity for Digraphs
3. Algorithmic Results on Cycle Rank
4. Applications in Formal Language Theory
5. Discussion

Outline

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Complexity Measures on Undirected Graphs

Important topic in algorithmic graph theory:

- Structural complexity restrictions can speed up algorithms
- Main result: many hard problems solvable in linear time on graphs with bounded treewidth.
- depending on application, also other measures interesting

What about Directed Graphs?

Lifting this successful theory to digraphs?

- Mixed success for treewidth: remarkable positive, but many negative algorithmic results

Here:

- Concentrate on measures whose theory can be lifted more easily: Separator number, pathwidth, cycle rank (=elimination tree height)
- Relations, algorithmic results, applications
- In particular: Cycle rank important for theory of regular expressions

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Cycle Rank: Definition

- Acyclic graph has $\mathrm{r}(G) = 0$
- If graph is a single SCC and $E \neq \emptyset$, then $\mathrm{r}(G) = \min_{v \in V} \mathrm{r}(G \setminus v) + 1$.
- Otherwise, we compute cycle rank for each component and take maximum

Observe: quite different from vertex feedback number

Intuition: Time needed to break all cycles "in parallel" by removing a single vertex from each component in each round.

Example

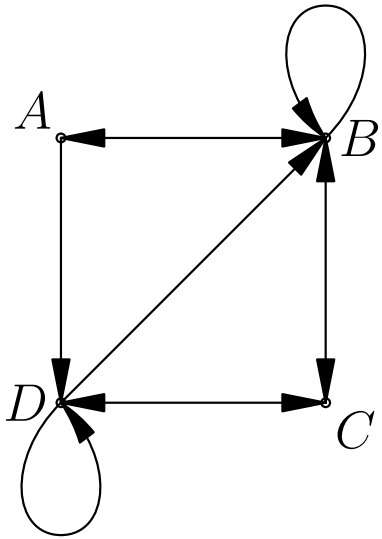
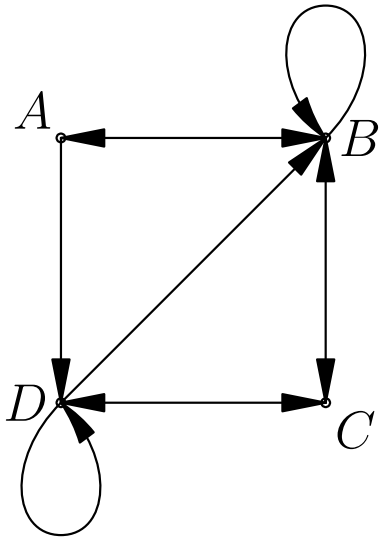


Figure: A recursion tree giving the cycle rank for a toy digraph.

Example



$$(A, \{A, \underset{\odot}{B}, C, D\})$$

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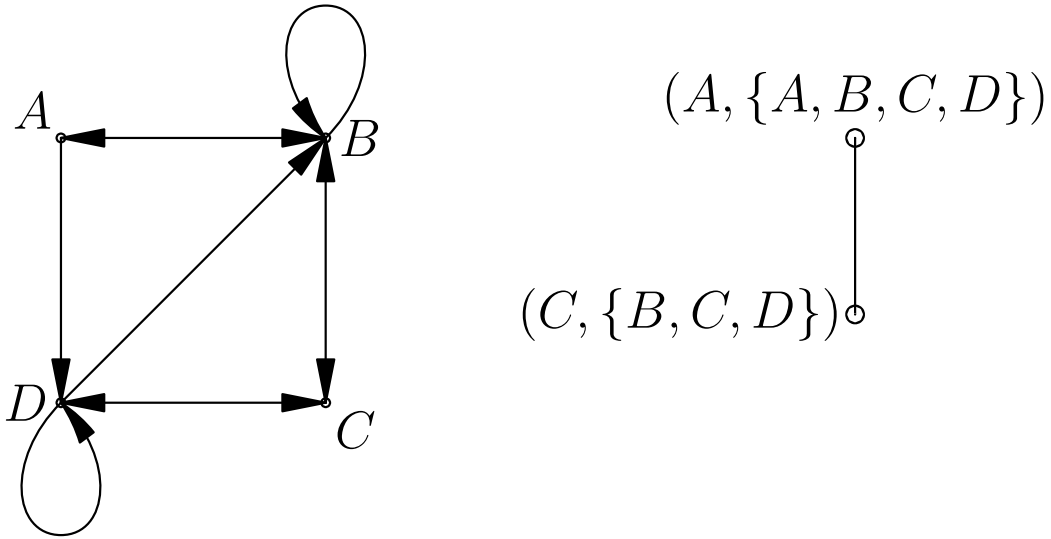

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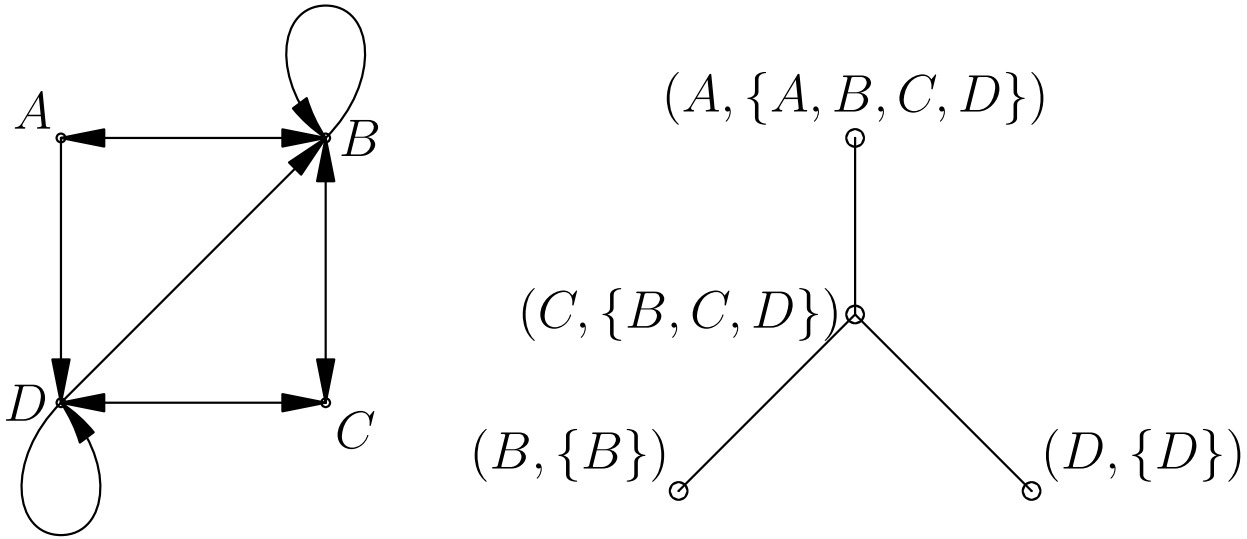

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Cycle Rank: Some historical notes

- Defined in 1960s by Eggan and Büchi.
  Motivation: formal languages
- remained largely unnoticed outside formal language community
- For undirected graphs rediscovered in 1980s as elimination tree height
  Motivation: Fast computation on sparse symmetric matrices
- Generalized to digraphs by Eisenstat and Liu in 2005
  Motivation: sparse matrices, again

Relation to other measures

We compared cycle rank other complexity measures
Result in short:

- the inspected measures admit a linear ordering
- Cycle rank is upper bound for all these measures
- $r(G)$ can exceed all other measures by at most logarithmic factor, i.e. $O(\log |V|)$.

Generalizes an influential result from undirected theory

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Complexity Results for Cycle Rank

Theorem

The decision version of determining the cycle rank of a given digraph G is **NP***-complete.* □

Hardness not unexpected, some comparable measures not known to be in **NP**!
How to cope with? Naive algorithm runs in time $O(n!)$.

Theorem

The cycle rank of an n-vertex graph can be determined in time and space $\widetilde{O}(2^n)$. □

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Cycle Rank and Regular Expressions

- Cycle rank closely related to star height of regular languages
- **NP**-completeness of cycle rank can be used to show:

Theorem

The star height problem for bideterministic finite automata is **NP***-complete.* □

General case, i.e. deterministic finite automaton input, largely open: **EXPSPACE** (Kirsten 2007) vs **NP**-hard (here)

Cycle Rank and Regular Expressions (2)

Another application:

- Predicting minimum regular expression size often difficult
- Can bound required size from below using star height (joint work with M. Holzer, presented at ICALP '08)
- Proof essentially relies on cycle rank

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Discussion

- Approximation algorithm for cycle rank? Would imply (a bit weaker) approximation for D-width, DAG-width, directed path-width, ...
- Better (exact/fixed-parameter) algorithms for cycle rank? In particular, presented algorithm uses exponential space
- What other results from undirected theory can be more easily lifted to directed graphs?