Characterizations of Branching Time Future Temporal Logics on Finite Trees

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Verification often uses logic to specify the expected behavior of systems.

Determining the expressive power of these logics is an important question.

In the case of strings, most known effective characterizations were obtained by the methods of algebraic theory of automata and semigroups.

Another method to obtain inexpressibility results is provided by the Ehrenfeucht-Fraïssé type games.

We are concerned with the definability problem of some classes of logics on (finite, ranked, ordered) trees.
Contents

- Notation and the logic CTL
- The logics FTL(ℒ)
- Algebraic characterization of FTL(ℒ)
- Game-theoretic characterization of FTL(ℒ)
A rank type $R$ is a finite set of nonnegative integers, containing $0$.

$\Sigma = \bigcup_{n \in R} \Sigma_n$: finite ranked alphabet of rank type $R$.

$T_\Sigma$ stands for the set of (variable-free) $\Sigma$-trees.

Tree language (over $\Sigma$): any set $L \subseteq T_\Sigma$.

$t|_x$ stands for the subtree of $t$ rooted at the node $x$. 
A (finite) $\Sigma$-tree automaton $\mathbb{A} = (A, \Sigma)$ consists of a (finite) state set $A$ and an elementary operation $\sigma^A : A^n \rightarrow A$ for each $\sigma \in \Sigma_n$.

Given also a set $A' \subseteq A$ of accepting states, $\mathbb{A}$ recognizes the language $L = \{ t : t^A \in A' \}$ with the set $A'$ of accepting states.

$L$ is regular if can be recognized by a finite tree automaton.

$A_L$ denotes the minimal recognizer of $L$ (unique up to isomorphism).
The logic CTL (Emerson, Clarke 1982)

Formulae of CTL over finite trees
- For any $\sigma \in \Sigma$, $p_\sigma$ is a formula.
- Whenever $\varphi$ and $\psi$ are formulae, then so are ($\neg \varphi$) and ($\varphi \lor \psi$).
- If $\varphi$ is a formula, EX($\varphi$) is also a formula.
- If $\varphi$ and $\psi$ are formulae, then EU($\varphi, \psi$) is a formula as well.

Semantics
A $\Sigma$-tree $t = \sigma(t_1, \ldots, t_n)$ satisfies $\varphi$, in notation $t \models \varphi$ iff one of the following conditions hold:
- $\varphi = p_\sigma$;
- Boolean connectives are handled as usual;
- $\varphi = \text{EX}(\varphi_1)$ and $t_i \models \varphi_1$ for some $i \in [n]$;
- $\varphi = \text{EU}(\varphi_1, \varphi_2)$ and there exists a node $x \in \text{dom}(t)$ such that
  - $t|_x \models \varphi_2$ and
  - for any ancestor $y$ of $x$, $t|_y \models \varphi_1$. 
The logic CTL (Emerson, Clarke 1982)

A formula \( \varphi \) defines the language \( \{ t \in T_\Sigma : t \models \varphi \} \).

The definability problem of CTL
Given a regular tree language \( L \) (say, by \( A_L \)), is \( L \) definable in CTL?

The decidability status of this problem is unknown.
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- Notation and the logic CTL
- The logics FTL(\(\mathcal{L}\))
- Algebraic characterization of FTL(\(\mathcal{L}\))
- Game-theoretic characterization of FTL(\(\mathcal{L}\))
The logic $\text{FTL}(\mathcal{L})$ (Ésik 2005)

**Syntax**

For a class $\mathcal{L}$ of tree languages, the formulae of $\text{FTL}(\mathcal{L})$ (over the alphabet $\Sigma$) are the following:

- for each $\sigma \in \Sigma$, $p_\sigma$ is a formula;
- whenever $\varphi$ and $\psi$ are formulae, then so are $(\lnot \varphi)$ and $(\varphi \lor \psi)$;
- if $L \in \mathcal{L}$ is a $\Delta$-tree language and for each $\delta \in \Delta$, $\varphi_\delta$ is a formula, then $L(\delta \mapsto \varphi_\delta)$ is also a formula.

**Semantics**

For any tree $t \in T_\Sigma$ and formula $\varphi$, let $t \models \varphi$ iff

- $\varphi = p_\sigma$ and $\text{Root}(t) = \sigma$;
- Boolean connectives are treated as usual;
- $\varphi = L(\delta \mapsto \varphi_\delta)$ and the characteristic tree of $t$ determined by the family $\{\varphi_\delta : \delta \in \Delta\}$ belongs to $L$. 

Semantics

Characteristic tree

The characteristic tree $\hat{t} \in T_\Delta$ of $t$ determined by $\{\varphi_\delta : \delta \in \Delta\}$ is a relabeling of $t$:
For any $n$-ary node $x$, let $\hat{t}(x) = \delta$ iff
- $t|_x \models \varphi_\delta$ and $\delta$ is the first such element of $\Delta_n$, OR
- $t|_x \not\models \bigvee_{\delta' \in \Delta_n} \varphi_{\delta'}$ and $\delta$ is the last element of $\Delta_n$.

(We assume that each alphabet $\Delta$ comes with a fixed linear ordering. This has no impact on the expressive power.)

A formula $\varphi$ over $\Sigma$ defines the tree language $L_\varphi = \{t \in T_\Sigma : t \models \varphi\}$. 
Many temporal logics, including CTL and the modular temporal logic can be expressed as $\text{FTL}(\mathcal{L})$ for some simple class $\mathcal{L}$ of regular tree languages.

The definability problem of $\text{FTL}(\mathcal{L})$ for a class $\mathcal{L}$ of (regular) tree languages

Given a (regular) tree language $L$, is $L$ definable in $\text{FTL}(\mathcal{L})$?

- The decidability status of the problem for CTL is still unknown.
- Positive results were given for some fragments of CTL (Ésik ’05, Bojańczyk and Walukiewicz ’06, Ésik and Iván ’08).
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- Game-theoretic characterization of FTL(\(\mathcal{L}\))
Let $\mathbb{A} = (A, \Sigma)$ and $\mathbb{B} = (B, \Delta)$ be two tree automata and $\alpha : A \times \Sigma \to \Delta$ a rank-preserving function. Then the **Moore product** of $\mathbb{A}$ and $\mathbb{B}$ determined by $\alpha$ is the tree automaton $\mathbb{A} \times_{\alpha} \mathbb{B} = (A \times B, \Sigma)$ with

$$
\sigma^{\mathbb{A} \times_{\alpha} \mathbb{B}} \left( (a_1, b_1), \ldots, (a_n, b_n) \right) = (\sigma^A(a_1, \ldots, a_n), \delta^B(b_1, \ldots, b_n))
$$

with $\delta = \alpha(\sigma^A(a_1, \ldots, a_n), \sigma)$ for each $n \in R$, $\sigma \in \Sigma_n$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$. 
Theorem [Ésik, Iván 2008]

Let $\mathcal{L}$ be a class of regular tree languages. Then a tree language $K$ is definable in $\text{FTL}(\mathcal{L})$ if and only if $K$ is regular and $A_K$ is contained in the least class $\mathbf{V}$ of finite tree automata satisfying each of the following conditions:

- $\mathbf{V}$ contains $A_L$ for each $L \in \mathcal{L}$;
- $\mathbf{V}$ contains a specific tree automaton $D_0$;
- $\mathbf{V}$ is closed under taking direct products, homomorphic images, subautomata and renamings (that is, a pseudovariety of finite tree automata);
- $\mathbf{V}$ is closed under the Moore product.

(If each quotient of each language in $L$ is definable in $\text{FTL}(\mathcal{L})$.)
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Let $\mathcal{L}$ be a class of tree languages, $n \geq 0$ an integer and $t_0$, $t_1$ be trees over some alphabet $\Sigma$. The $n$-round FTL($\mathcal{L}$)-game on $(t_0, t_1)$ is played between Spoiler and Duplicator as follows:

- If $\text{Root}(t_0) \neq \text{Root}(t_1)$, Spoiler wins.
- Otherwise, if $n = 0$, Duplicator wins.
- Otherwise, Spoiler picks a language $L \in \mathcal{L}$ over an alphabet $\Delta$;
  - $\Delta$-relabelings $\hat{t}_0$ and $\hat{t}_1$ of $t_0$ and $t_1$, resp., such that exactly one of them is in $L$.
  - Duplicator picks two nodes, $x$ of $t_i$ and $y$ of $t_j$ ($i = j$ is allowed) with $\hat{t}_i(x) \neq \hat{t}_j(y)$.
- An $(n - 1)$-round FTL($\mathcal{L}$)-game is played on $(t_i|_x, t_j|_y)$. Whoever wins the subgame, wins also the game.
Game-theoretic characterization of FTL($\mathcal{L}$)

**Theorem [Ésik, Iván 2008]**

For any class $\mathcal{L}$ of tree languages, integer $n \geq 0$ and trees $s, t$ over an alphabet $\Sigma$, the following are equivalent:

1. Duplicator has a winning strategy in the $n$-round FTL($\mathcal{L}$)-game on $(s, t)$;
2. $s$ and $t$ agree on all FTL($\mathcal{L}$)-formulae over $\Sigma$ with nesting depth at most $n$.

**Corollary**

Suppose $\mathcal{L}$ is a finite class of tree languages. The following are equivalent for any $\Sigma$-tree language $L$:

1. $L$ is definable in FTL($\mathcal{L}$);
2. there exists an $n \geq 0$ such that Spoiler has a winning strategy in the $n$-round FTL($\mathcal{L}$)-game over any pair $(s, t)$ of $\Sigma$-trees with $s \in L$, $t \notin L$. 


Application for $\textit{TL}[\textit{EF}^\ast]$ 

Syntax

The formulae of $\textit{TL}[\textit{EF}^\ast]$ over $\Sigma$ are given as follows:

- for each $\sigma \in \Sigma$, $p_\sigma$ is a formula;
- whenever $\varphi$ and $\psi$ are formulae, so are $(\neg \varphi)$ and $(\varphi \lor \psi)$;
- whenever $\varphi$ is a formula, so is $\textit{EF}^\ast(\varphi)$.

Semantics

The tree $t$ satisfies $\textit{EF}^\ast(\varphi)$ if $t|_{x} \models \varphi$ for some node $x$ of $t$.

Observe that $\textit{EF}^\ast(\varphi)$ is a shorthand for $\textit{EU}(\top, \varphi)$, thus $\textit{TL}[\textit{EF}^\ast]$ is a syntactic fragment of $\textit{CTL}$.
Characterization of $TL[EF^*]$ 

**Theorem (for binary trees) [Ésik, Iván 2008]**

Let $R = \{0, 2\}$. Then $L$ is definable in $TL[EF^*]$ iff $L$ is regular and $\mathbb{A}_L$ satisfies the following identities:

- $\sigma(x, y) = \sigma(y, x)$;
- $\sigma(x, y) = \sigma(x, x)$, whenever $y \preceq x$;
- $\sigma(x, y) = \sigma(x', y')$, whenever $x \sim x'$ and $y \sim y'$;
- $\sigma(x, \sigma(x, y)) = \sigma(x, y)$.

Here

- $x \preceq y$ iff the state $y$ is accessible from the state $x$;
- $x \sim y$ iff $x \preceq y$ and $y \preceq x$. 
We defined the Moore product of tree automata providing an algebraic characterization of $\text{FTL}(\mathcal{L})$.

We defined the $n$-round $\text{FTL}(\mathcal{L})$-game providing a game-theoretic characterization of $\text{FTL}(\mathcal{L})$.

We showed the decidability of the definability problem of some fragments of CTL, e.g. that of $\text{TL}[\text{EF}^*]$.

Thank you for your attention.